

# Discrete Holomorphicity at Two-Dimensional Critical Points

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Received: 24 July 2009 / Accepted: 4 November 2009 / Published online: 13 November 2009  
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**Abstract** After a brief review of the historical role of analyticity in the study of critical phenomena, an account is given of recent discoveries of discretely holomorphic observables in critical two-dimensional lattice models. These are objects whose correlation functions satisfy a discrete version of the Cauchy-Riemann relations. Their existence appears to have a deep relation with the integrability of the model, and they are presumably the lattice versions of the truly holomorphic observables appearing in the conformal field theory (CFT) describing the continuum limit. This hypothesis sheds light on the connection between CFT and integrability, and, if verified, can also be used to prove that the scaling limit of certain discrete curves in these models is described by Schramm-Loewner evolution (SLE).

**Keywords** Integrable models · Conformal field theory · Schramm-Loewner evolution

## 1 Introduction

Analyticity has played a key role in the development of our mathematical understanding of the nature of critical phenomena. At various levels, the assertion of the analytic dependence of a suitable physical observable on some variable has provided a powerful starting point for theories, and its ultimate failure has informed the next step in their refinement.

The first example of this was perhaps Landau theory, in which the free energy  $F[\mathbf{M}]$  is postulated to be analytic in the order parameter  $\mathbf{M}$ :

$$F[\mathbf{M}] = \int \left( (\nabla \mathbf{M})^2 + \text{Tr}(\mathbf{H} \cdot \mathbf{M}) + r_0 \text{Tr} \mathbf{M}^2 + \lambda_3 \text{Tr} \mathbf{M}^3 + \dots \right) d^d x.$$

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In this expression, all terms allowed by symmetry are in principal present. The requirement that  $\mathbf{M}$  should minimize  $F$ ,  $\delta F/\delta \mathbf{M} = 0$ , implies that there are critical points (in the mathematical sense) at certain values of the parameters. The singular behavior close to these is described by the mathematical theory of bifurcations. The critical exponents which emerge from this analysis are super-universal in that they do not depend on the spatial dimension  $d$  or the symmetry of the order parameter.

Of course, the successes and the limitations of Landau theory are well understood. It does not account for the fluctuations in the order parameter. In high enough dimensions these can usually be ignored as far as the universal behavior is concerned, but in general they should be taken into account by computing the full Landau-Ginzburg-Wilson partition function

$$Z[\mathbf{H}] = \int_{|q| < \Lambda} [d\mathbf{M}(x)] e^{-F[\mathbf{M}]},$$

with a suitable cut-off  $\Lambda$  on the short wavelength modes. This of course is impossible except in some simple cases, but these can often be used as a starting point for a perturbative expansion. However this approach generally fails near the critical point.

The renormalization group (RG) of Wilson, Fisher and others was a way of dealing with this problem, by considering the response to changes in the cut-off  $\Lambda \rightarrow \Lambda e^{-\ell}$  in such a way as to move the parameters into a region where perturbation theory is applicable. Requiring that the long-distance physics remains the same then leads to the RG flow equations

$$\frac{d\lambda_j}{d\ell} = -\Lambda \frac{\partial \lambda_j}{\partial \Lambda} = -\beta_j(\{\lambda\})$$

for the parameters  $\{\lambda_j\}$  in  $F[\mathbf{M}]$ . The critical behavior is then controlled by the fixed points where  $\beta_j(\{\lambda\}) = 0$ . One of the crucial assumptions of the RG theory is that these functions themselves are analytic in the  $\{\lambda_j\}$  near the fixed point. Thus analyticity is moved to a higher level of abstraction: the non-analyticity of the free energy results from a potentially infinite number of applications of the analytic RG mapping. (Interestingly enough there is very little direct evidence for the strict validity of this assumption. It is largely based on analysis of perturbation theory which begs the question. Solvable models in two dimensions have so much analytic structure built into them that analyticity of the RG flows is almost inevitable, and possible detailed violations of analyticity have not, to my knowledge, been carefully studied in numerical simulations in higher dimensions. Indeed in one case,  $d = 0$ , when the beta-function can be evaluated analytically, it displays an essential singularity at the fixed point.)

The next example of the use of analyticity is in the solution of integrable models in two dimensions. In this case the independent variable parameterizes the solution manifold on which the Boltzmann weights satisfy the Yang-Baxter equations. Its physical interpretation, to be discussed later, is the degree of anisotropy of the model in two-dimensional euclidean space. The Yang-Baxter equations imply that the row-to-row transfer matrices at different values of this parameter commute. Assuming that the analytic properties of the local weights in the parameter lift to thermodynamic quantities, Baxter and others have shown that these obey functional relations which often determine them completely.

The other main approach to understanding two-dimensional critical behavior attempts to describe the continuum critical scaling limit directly, and also exploits analyticity. This is the approach of conformal field theory (CFT), more recently also linked to Schramm-Loewner evolution (SLE) [1, 2]. In this case the analyticity is in the two-dimensional coordinates  $z = x + iy$  and  $\bar{z} = x - iy$ . In CFT, correlation functions of certain observables are holomorphic functions of  $z$  (or antiholomorphic functions of  $\bar{z}$ ). These observables are of two

types: conserved currents corresponding to continuous symmetries, such as the stress tensor ( $T(z)$ ,  $\bar{T}(\bar{z})$ ), or so-called parafermions  $\psi_s(z)$  whose 2-point correlation function in the full plane have the form

$$\langle \psi_s(z_1) \psi_s(z_2) \rangle \sim (z_2 - z_1)^{-2s},$$

where  $s$ , the conformal spin, is in general fractional.

These holomorphic (and antiholomorphic) objects are the building blocks of many CFTs. They also give an explicit meaning to the statement of conformal covariance: if we consider the scaling limit of a critical lattice model in a domain  $\mathcal{D}$ , then the behavior of a multi-point holomorphic correlator is completely determined by its singular behavior at coincident points and the boundary conditions on  $\partial\mathcal{D}$ . If these are themselves conformally covariant (e.g. if  $\lim_{z \rightarrow \partial\mathcal{D}} \arg \psi_s(z)$  is determined by the boundary tangent angle), then, under any conformal mapping  $f : \mathcal{D} \rightarrow \mathcal{D}'$  of the interior of  $\mathcal{D}$  to another domain  $\mathcal{D}'$

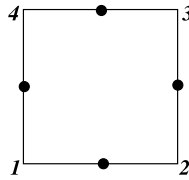
$$\langle \psi_s(z_1) \psi_s(z_2) \rangle_{\mathcal{D}} = f'(z_1)^s f'(z_2)^s \langle \psi_s(z'_1) \psi_s(z'_2) \rangle_{\mathcal{D}'}$$

It is the purpose of this paper to review recent work which forges a link between these last two realizations of analyticity. More specifically, starting from certain lattice models we identify so-called discretely holomorphic observables, whose correlators satisfy a lattice version of the Cauchy-Riemann equations. These have fractional spin by construction, and are presumably the lattice precursors of the parafermions in the corresponding CFT. We find, as expected, that discrete holomorphicity holds only when the Boltzmann weights lie on the critical manifold of the model, but, more surprisingly, that they also lie on the *integrable* critical manifold, that is, they satisfy the Yang-Baxter equations.

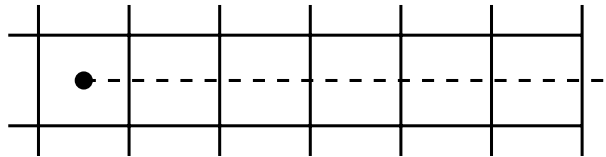
This work has been described in detail in several papers. In [3–5] a discretely holomorphic observable was identified for the random cluster representation of the  $q$ -state Potts model. In [6] several were found, directly from the Boltzmann weights, for the  $Z_N$  clock models, and the close connection to integrability was first observed. Smirnov [4, 5] also found such an observable for the random curve representation of the  $O(n)$  model on the honeycomb lattice. This was extended to the  $O(n)$  model, and further generalizations, on the square lattice in [7], where the parameter space is large enough to make clear the connection to integrability. Smirnov's objective in identifying these holomorphic objects associated with curves was to show that their scaling limit is described by Schramm-Loewner evolution (SLE). This approach is fully described in [4, 5] and we refer the reader there. Suffice it to say that proving that the discretely holomorphic lattice observables go over into fully holomorphic quantities in the scaling limit is in general a very difficult problem which has been solved completely only in a few special cases, in particular that of the Ising model ( $q = 2$  or  $n = 1$ ) [4, 5].

In this account we shall describe in detail only two examples. The first is the simplest: the case  $N = 2$  of the  $Z_N$  model, more usually known as the Ising model. In this case the parafermion has conformal spin  $s = \frac{1}{2}$  and is related to the fermionic objects used in the various exact solutions of the model, going back to Onsager. It is defined, in a standard way, as a product of nearby order and disorder variables. This identification dates back (at least) to Fradkin and Kadanoff [8], but the simple argument given in [6] that a particular lattice definition is discretely holomorphic is, to our knowledge, new. An essential ingredient in the definition of a disorder operator is that the model must possess a dual description in the sense of Kramers and Wannier. However, the second example is the  $O(n)$  model which does not possess this. In this case the parafermionic observables are instead defined in terms of the random curve representation of the model, which is nonlocal in terms of the original spin degrees of freedom.

**Fig. 1** A face of the square lattice. The discretely holomorphic function  $F(z)$  is defined at the *mid-points* of the edges



**Fig. 2** The string defining a disorder operator



Before proceeding, however, we give a proper definition of discrete holomorphicity. Suppose  $\mathcal{G}$  is a planar graph embedded in  $\mathbf{R}^2$ , for example a square lattice. Let  $F(z_{ij})$  be a complex-valued function defined at the midpoints  $z_{ij}$  of each edge  $(ij)$ . Then  $F$  is discretely holomorphic on  $\mathcal{G}$  if

$$\sum_{(ij) \in \mathcal{F}} F(z_{ij})(z_j - z_i) = 0, \tag{1}$$

where the sum is over the edges of each face  $\mathcal{F}$  of  $\mathcal{G}$ . This is a discrete version of the contour integral. For a square lattice (see Fig. 1) it reduces to

$$F(z_{12}) + iF(z_{23}) + i^2F(z_{34}) + i^3F(z_{41}) = 0.$$

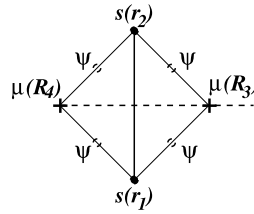
A little thought shows, however, that the total number of such equations, one for each face of  $\mathcal{G}$ , is in general far less than the number of unknowns, one for each edge. Therefore even on the lattice this system does not determine  $F(z_{ij})$  for suitable boundary conditions, unless further information is available. (For the Ising model it turns out that more is known about the phase of  $F$ , so the system is rigid.) However, in the continuum limit as the lattice spacing tends to zero, we can approximate arbitrarily closely the contour integral  $\int_C F(z) dz$  by a sum of the left-hand side of (1) over faces  $\mathcal{F}$  which tile the interior of  $C$ , and therefore assert that it vanishes for all reasonable contours  $C$ . Morera’s theorem then assures us that if  $F(z)$  is continuous then it is analytic. In field theory the continuity of correlation functions is usually taken for granted, but strictly this needs to be proved.

### 2 The Ising and $Z_N$ Models

The square lattice Ising model has spins  $s(r) = \pm 1$  at the vertices  $r$ . The Boltzmann weights are  $\exp(-\mathcal{H})$  with  $\mathcal{H} = -\sum_{rr'} J_{rr'} s(r)s(r')$ , where the sum is over edges  $(rr')$ . Note that the weights can also be written  $\prod_{rr'} (1 + (\tanh J_{rr'}) s(r)s(r'))$ . The insertion of a disorder operator  $\mu(R)$  at the dual vertex  $R$  corresponds to changing  $J_{rr'} \rightarrow -J_{rr'}$  on all edges which cross a ‘string’ attaching  $R$  to the boundary, see Fig. 2. That is

$$\mu(R) = \prod_{(rr') \ddagger \text{string}} \frac{1 - (\tanh J_{rr'}) s(r)s(r')}{1 + (\tanh J_{rr'}) s(r)s(r')}. \tag{2}$$

**Fig. 3** Elementary square on whose edges the parafermionic observables are defined



The correlator  $\langle \mu(R_1)\mu(R_2) \rangle$  corresponds to such a string connecting  $R_1$  and  $R_2$ , and is invariant under deformations of the string.

The parafermionic variables  $\psi_s(rR)$  are defined on the midpoints of each edge ( $rR$ ) connecting the vertex  $r$  to a neighboring dual vertex  $R$ , which form the covering lattice:

$$\psi_s(rR) = s(r) \cdot \mu(R) e^{-is\theta(rR)}.$$

Here  $\theta(rR)$  is the angle that ( $rR$ ) makes with (say) the positive  $x$ -axis, but we have to be careful about its multivaluedness (see below).

Consider now an elementary square (see Fig. 3). From (2) we have

$$(1 + (\tanh J_y)s(r_1)s(r_2)) \mu(R_4) = (1 - (\tanh J_y)s(r_1)s(r_2)) \mu(R_3), \tag{3}$$

where we have set  $J_{rr'} = J_{x,y}$  depending on whether ( $rr'$ ) is parallel to the  $x$  or  $y$ -axis. Now multiply (3) by  $s(r_1)$  and  $s(r_2)$  and use  $s(r)^2 = 1$ . This leads to two linear equations in the four parafermionic variables defined on the edges of the square. Simple algebra then shows that these imply the discrete holomorphicity condition (1) as long as:

- we distort each square into a rhombus whose angle depends on the anisotropy  $J_y/J_x$ ;
- we are careful, since  $s = \frac{1}{2}$  is non-integer, to define  $\theta(rR)$  consistently so that it varies only by increments in the interval  $(-\pi, \pi)$  on going around the square;
- the couplings lie on the critical manifold  $\sinh J_x \sinh J_y = 1$ .

This example illustrates the general results discussed in the introduction. It also shows how the lattice should be embedded in  $\mathbf{R}^2$  so that its continuum limit is rotationally and conformally invariant. However since the nearest neighbor Ising model is integrable for all values of the couplings ( $J_x, J_y$ ), it is not general enough to illustrate the role of integrability.

This is afforded by the generalization to the  $Z_N$  models [6]. In these models the Ising spins are generalized to complex roots of unity such that  $s(r)^N = 1$ . The Boltzmann weights take the form

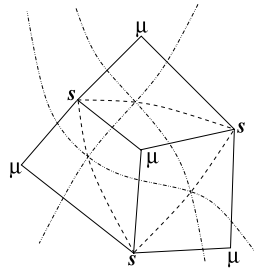
$$\prod_{rr'} \left( 1 + \sum_{k=1}^{N-1} x_{rr'}^{(k)} (s(r)^* s(r'))^k + \text{c.c.} \right).$$

It is then found that one can identify discretely holomorphic parafermions, again the product of neighboring order and disorder operators, with spins

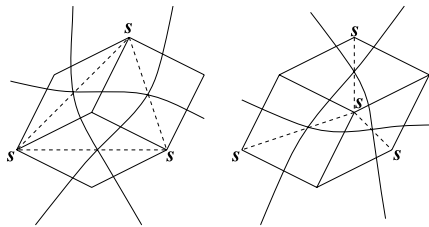
$$s = \frac{k(N - k)}{N} \quad (1 \leq k \leq N/2),$$

as long as the parameters lie on the manifold

$$x_x^{(k)}(\alpha) = \prod_{j=0}^{k-1} \frac{\sin((\pi j + \alpha)/N)}{\sin((\pi(j + 1) - \alpha)/N)} \tag{4}$$



**Fig. 4** Part of a Baxter lattice. The faces of the graph  $\mathcal{L}$  formed by the dash-dot lines are 2-colorable (not shown). Order variables  $s$  and disorder variables  $\mu$  are associated with alternately colored faces respectively. The nearest neighbor interactions between the order variables  $s$  are indicated by dashed lines. The covering lattice is shown as solid lines. The theorem of Kenyon and Schlenker [12] asserts that for every such graph the covering lattice admits a rhombic embedding in the plane, that is one where all its edges have the same length



**Fig. 5** Two different tilings of a hexagon by the same set of three rhombi. The right-hand case has an additional vertex associated to an order operator  $s$  as compared to that on the left. Discrete holomorphicity for each rhombus fixes the couplings on the dashed lines to be related by the star-triangle transformation. The two pictures are also related in the original graph  $\mathcal{L}$  by moving one of the curves past the vertex formed by the other two—the Yang-Baxter relation

and  $x_y^{(k)}(\alpha) = x_y^{(k)}(\pi/2 - \alpha)$ . Here  $\alpha$  is the half-angle at the vertex of the rhombus into which the square lattice must be distorted, and therefore measures the degree of anisotropy. However this manifold is precisely the critical integrable case for the general nearest neighbor  $Z_N$  model found by Fateev and Zamolodchikov [9]. The values of the conformal spins above agree precisely with those of the parafermionic holomorphic conformal fields in the CFT postulated by the same authors [10, 11] to describe the continuum limit of this model.

The above analysis can be generalized simply to a general graph  $\mathcal{L}$  whose faces are 2-colorable, sometimes called a Baxter lattice—see Fig. 4. The  $Z_N$  spins  $s(r)$  are defined on the faces of a given color, and the disorder variables  $\mu(R)$  on the others. Neighboring order and disorder operators then lie at the vertices of quadrilaterals which tile the plane. A theorem due to Kenyon and Schlenker [12] asserts that, under rather general conditions, they can be distorted in the plane so they are all rhombi, that is, they form an isoradial lattice, on which nearest neighbors are all the same distance apart. If now the interactions across each rhombus are chosen to satisfy (4), where  $\alpha$  is the half-angle of the rhombus, then the associated parafermion is discretely holomorphic on the isoradial lattice. Moreover the corresponding weights satisfy the star-triangle, or Yang-Baxter equations, as explained in Fig. 5.

We note that although we have defined the parafermions in this case directly in terms of the modified Boltzmann weights, they may also be identified with observables of ran-

dom curves in the model. For the case of the Ising model it is well known that the high-temperature expansion, in powers of  $\tanh J_{x,y}$ , of the correlation function  $\langle s(r)s(0) \rangle$  can be expressed as sum over graphs on the lattice. The argument is clearer for the honeycomb lattice, when these graphs consist of non-intersecting closed loops and one open curve  $\gamma$  from 0 to  $r$ . The additional complications on the square lattice are believed to be irrelevant to the scaling limit. The introduction of the disorder operator  $\mu(R)$  neighboring  $s(r)$  acts to weight configurations of the open curve with  $(-1)^N$  where  $N$  is the number of times  $\gamma$  crosses the string. This can also be written as  $e^{-is\theta_{0r}}$  where  $\theta_{0r}$  is the winding angle of  $\gamma$ . In the scaling limit this takes arbitrarily large values and so it does not matter whether we compute it in increments of  $\pm 2\pi$  as it crosses the string, or simply in increments of  $\pm \frac{\pi}{2}$  as we proceed along  $\gamma$ . This latter definition corresponds to the one used by Smirnov [4, 5].

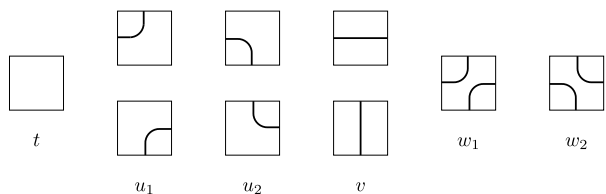
### 3 The $O(n)$ Model

As a second example we consider the so-called  $O(n)$  model on the square lattice, first considered by Nienhuis [13]. Although it can be written in terms of  $n$ -component spins  $s_a(r)$  ( $a = 1, 2, \dots, n$ ) located on the edges of a square lattice through local, but non-nearest neighbor, interactions, it is more easily formulated in terms of a gas of dilute non-intersecting planar loops. Each elementary square can take one of the configurations shown in Fig. 6, with the indicated weights. When patched together these curves form closed loops, each of which receives an additional weight  $n$ , which no longer has to be an integer and may be parameterized by  $n = -2\cos 2\eta$  with  $0 \leq \eta \leq \frac{\pi}{2}$ . Note that we have grouped the anisotropic weights so that there is symmetry under reflections in the diagonal axes. This symmetry is preserved when the plaquettes are deformed into rhombi. Since every loop configuration has an even number of plaquettes of type  $u_1$  or  $u_2$ , the change  $(u_1, u_2) \rightarrow (-u_1, -u_2)$  does not affect the Boltzmann weights.

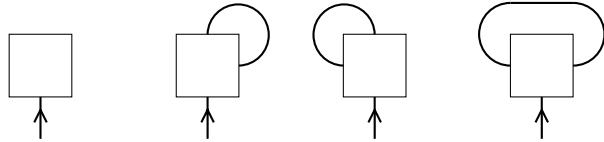
In order to define a suitable candidate for the parafermionic observable we must consider not only closed loops but also open curves which begin and end at different points, say 0 and  $r$ . Such configurations arise, as in the previous section, if we compute the correlator  $\langle s_a(r)s_a(0) \rangle$  of the  $O(n)$  spins. Once again, to define an object with non-zero spin, we additionally weight each open curve by a phase factor  $e^{-is\theta_{0r}}$ , where  $\theta_{0r}$  is the winding angle from 0 to  $r$ , the accumulation of the turns through  $\pm \frac{\pi}{2}$ . Note that this can be arbitrarily large. This type of observable for the  $O(n)$  model was first considered on the honeycomb lattice by Smirnov [4, 5].

Now consider the contributions to the holomorphicity equation (1) where  $r$  is one of the edges of an elementary square. Since the open curve ends at  $r$ , it must first intersect the square on some edge  $r'$  (not necessarily the same as  $r$ ). The configurations can then be decomposed according to the different possible  $r'$ . Without loss of generality we take it to be on the lower edge in Fig. 7. The configurations can then be further decomposed into classes corresponding to the various cases shown in this figure. In the last three cases a

**Fig. 6** Vertices of the  $O(n)$  loop model on the square lattice



**Fig. 7** Loop configurations with one edge of the face  $\mathcal{F}$  connected to point 0



curve connects two of the other edges: it may be part of a closed loop or of the open curve. The idea is to sum over all possible ways of connecting up the curves internally through the square, keeping the external configuration fixed, and to try to satisfy (1) for each of these configurations. This can be done, and yields the following linear system for the weights:

$$t + \mu u_1 - \mu \lambda^{-1} u_2 - v = 0, \tag{5}$$

$$-\lambda^{-1} u_1 + n u_2 + \lambda \mu v - \mu \lambda^{-1} (w_1 + n w_2) = 0, \tag{6}$$

$$n u_1 - \lambda u_2 - \mu \lambda^{-2} v + \mu (n w_1 + w_2) = 0, \tag{7}$$

$$-\mu \lambda^{-2} u_1 + \mu \lambda u_2 + n v - \lambda^{-2} w_1 - \lambda^2 w_2 = 0, \tag{8}$$

where we have set  $\lambda = e^{i\pi s}$ ,  $\varphi = (s + 1)\alpha$ ,  $\mu = e^{i\varphi}$ . For real weights, (5)–(8) are four complex linear equations for six real unknowns  $(t, u_1, u_2, v, w_1, w_2)$ , and we have the relations:

$$\text{Im} [(n + 1) \{5\} - \lambda \mu^{-1} \{6\} + \mu^{-1} \{7\}] = 0,$$

$$\text{Im} [\lambda \mu^{-1} (\lambda^2 - n \lambda^{-2}) \{6\} + \mu^{-1} (n \lambda^2 - \lambda^{-2}) \{7\} - (n^2 - 1) \{8\}] = 0.$$

Thus, we can generally reduce (5)–(8) to a  $6 \times 6$  real system.

There are two classes of solutions, for vanishing and non-vanishing  $v$ . First, if  $v = 0$ , then the configurations corresponding to {8} never occur, and so this equation does not hold. In the special case  $n = 1$ , there exists a non-trivial solution for any value of  $s$ :

$$t = \sin \pi s, \quad u_1 = \sin(\varphi - \pi s), \quad u_2 = \sin \varphi, \quad w_1 + w_2 = \sin \pi s. \tag{9}$$

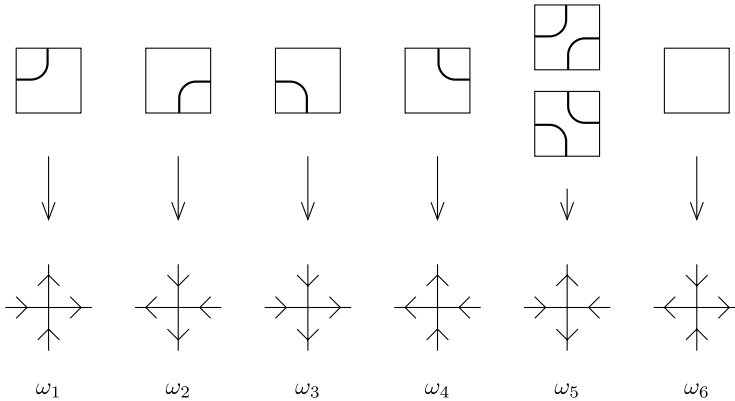
It turns out [7] that this model can be mapped onto the six-vertex model (see Fig. 8), with weights  $\omega_1 = \omega_2 = \sin(\varphi - \pi s)$ ,  $\omega_3 = \omega_4 = \sin \varphi$ ,  $\omega_5 = \omega_6 = \sin \pi s$ . The corresponding anisotropy parameter is  $\Delta = \cos \pi s$ . This is an example of a model admitting a holomorphic observable on the lattice, but for which the scaling limit of the corresponding curve cannot be described by simple SLE. This is because, for ordinary SLE, the central charge of the CFT is directly related to the SLE parameter  $\kappa$  [1, 2] and hence to the conformal spin  $s$ :  $c = 2s(5 - 8s)/(2s + 1)$ . In the present case, since the boundary conditions for the six-vertex model are not twisted, its scaling limit has central charge  $c = 1$  for all  $\Delta$ . However the conformal spin  $s$  varies continuously with  $\Delta$ . Therefore the scaling limit of the curve can be SLE, with  $\kappa = 4$ , for at most one value (in fact  $\Delta = 1/\sqrt{2}$ ). We conjecture that other values of  $\Delta$  in fact correspond to a variant of SLE called SLE(4,  $\rho$ ).

For  $v = 0$ ,  $n \neq -1$ , we get a  $5 \times 5$  linear system, with determinant  $(n^2 - 1)^2 \sin \varphi \sin(\varphi - \pi s)$ . Imposing  $\sin \varphi = 0$  yields  $(s + 1)\alpha = m\pi$  and in turn  $s = m'$ , where  $m, m'$  are integers. Thus, for the solution to exist at any value of  $\alpha$ , we have to set  $s = -1$ . The Boltzmann weights are then:

$$t = -u_1 - u_2, \quad w_1 = -u_1, \quad w_2 = -u_2. \tag{10}$$

The solution of the case  $\sin(\varphi - \pi s) = 0$  is similar, and leads to the same Boltzmann weights and spin  $s = -1$ . If we change the sign of  $u_1, u_2$ , then the model (10) is equivalent to a





**Fig. 8** Mapping of the  $O(n)$  model onto the six-vertex model for  $n = 1, v = 0$

dense loop model which corresponds to the critical  $q$ -state Potts model, with weight per loop  $\sqrt{q} = n + 1$ . To see this, fill empty spaces with loops of weight 1 (*ghost loops*). The local weights do not depend on the type of loops involved (actual or ghost loops), so each loop has an overall weight  $n + 1$ . As a consequence, the dense loop model has a lattice antiholomorphic observable ( $s < 0$ ), besides the holomorphic one found in [3]. However, several arguments rule out the hypothesis that this corresponds to an antiholomorphic field in the continuum limit. First,  $\psi_{s=-1}(z)$  is lattice antiholomorphic for any  $Q > 0$ , whereas it is well known that the self-dual Potts model is only critical for  $0 \leq Q \leq 4$ . Furthermore, the ratio  $u_1/u_2$  in (10) does not depend on the angle  $\alpha$ , which means that the same model has an antiholomorphic observable for any deformation angle: this is not acceptable physically in the continuum limit. So we conclude that, in the case  $v = 0$  and generic  $n \neq -1$ , the holomorphicity conditions (1) for the dilute  $O(n)$  model merely lead to the case of the dense loop model, but the corresponding  $\psi_s(z)$  is not a candidate for an antiholomorphic field in the continuum limit.

Let us now discuss the solutions of second class ( $v \neq 0$ ), for a generic value of  $n$ . We get the  $6 \times 6$  real system:

$$(\text{Re}\{5\}, \text{Re}\{6\}, \text{Im}\{6\}, \text{Re}\{7\}, \text{Im}\{7\}, \text{Re}\{8\}),$$

with determinant:  $(n^2 - 1) \sin \varphi \sin(\varphi - \pi s)(2 \cos 4\pi s - 3n + n^3)$ . Non-trivial solutions exist if the spin satisfies:

$$\cos 4\pi s = \cos 6\eta. \tag{11}$$

The various solutions to (11) can be parameterized by extending the range of  $\eta$  to  $[-\pi, \pi]$ , and setting:

$$s = \frac{3\eta}{2\pi} - \frac{1}{2}. \tag{12}$$

Then, we get the second class of solutions, with Boltzmann weights:

$$t = -\sin(2\varphi - 3\eta/2) + \sin 5\eta/2 - \sin 3\eta/2 + \sin \eta/2, \tag{13}$$

$$u_1 = -2 \sin \eta \cos(3\eta/2 - \varphi), \tag{14}$$

$$u_2 = -2 \sin \eta \sin \varphi, \quad (15)$$

$$v = -2 \sin \varphi \cos (3\eta/2 - \varphi), \quad (16)$$

$$w_1 = -2 \sin(\varphi - \eta) \cos (3\eta/2 - \varphi), \quad (17)$$

$$w_2 = 2 \cos (\eta/2 - \varphi) \sin \varphi. \quad (18)$$

A remarkable fact is that the weights (13)–(18) are a solution of the Yang-Baxter equations for the  $O(n)$  loop model on the square lattice. Indeed, after a change of variables  $\varphi \rightarrow \psi + (\pi + \eta)/4$ , they coincide with the integrable weights in [13]. So, by solving the holomorphicity equations (5)–(8) on a deformed lattice, we recover the integrable weights.

Other more complicated loop models (for example one with different types of loops known as the C2(1) model [14]) can also be studied with similar results [7].

#### 4 Conclusions and Further Remarks

We have given two main examples of lattice models, the  $Z_N$  model and the  $O(n)$  model on a square lattice, in which observables can be identified whose correlators are discretely holomorphic, as long as the weights are both critical and satisfy the Yang-Baxter relations. This is surprising, since the holomorphicity conditions are linear in the weights, and work for a fixed value of the anisotropy parameter, while the Yang-Baxter relations are cubic functional equations for the weights. While in the case of the  $Z_N$  models the connection to integrability may be understood by generalizing the problem to an inhomogeneous Baxter lattice, this explanation is at present missing for the loop models. While it would be nice to elevate these observations to a more general result connecting holomorphicity, integrability and conformal field theory, the counter-example given in Sect. 3 in which a lattice holomorphic observable apparently does not correspond to a conformal field should warn us that there may be subtleties. For the  $Z_N$  models for larger values of  $N$ , problems in a lattice identification of the value of the conformal spin  $s$  were also noted in [6].

Although it is to be hoped that the results of Smirnov [4, 5] in using these holomorphic observables to prove that the scaling limit of lattice curves is given by SLE can be extended to other models, the examples we have given which correspond to CFTs with central charge  $c \geq 1$  show that this may not always be the case.

Finally, it is to be hoped that at some Statistical Mechanics Conference in the future the correct extension of these ideas to higher dimensions will be announced!

**Acknowledgements** I would like to thank my collaborators Valentina Riva, Mohammed Rajabpour and Yacine Ikhlef, as well as Stas Smirnov and Paul Fendley, for many discussions on this subject. This work was supported in part by EPSRC grant EP/D050952/1.

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